DOCUMENT RESUME

ED 173 134 **SE 027 967**

AUTHOR	Anderson, R. D.; And Others
TITLE	Mathematics for Junior High School. Supplementary Units. Commentary for Teachers.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	60
NOTE	62p.; For related documents, see SE 027 963-966 and ED 143 530; Contains occasional light and broken type
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	Curriculum; *Curriculum Guides; Decimal Fractions; *Geometry; *Instruction; Mathematics Education; Number Concepts; *Prime Numbers; Secondary Education; *Secondary School Mathematics; *Set Theory
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is a supplementary manual for teachers using SMSG junior high school text materials. A chapter-by-chapter commentary on the text is given as well as answers to all the exercises. Chapter topics include: (1) sets; (2) projective geometry; (3) repeating decimals; (4) tests for divisibility; (5) finite differences; and (6) prime numbers. (MP)

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

SUPPLEMENTARY UNITS

Commentary for Teachers

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

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ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

COMMENTARY FOR TEACHERS

TABLE OF CONTENTS

	Page
Supplementary Unit 1 - Sets	1
1-1 Introduction	1
1-2 Sets, Their Number and Their Subsets	2
1-3 Operations with Sets	4
1-4 Order, One-to-One Correspondence, the Number of a Set, and Counting	7
Supplementary Unit 2 - Special Figures in Projective Geometry	11
Supplementary Unit 3 - Repeating Decimals and Tests for Divisibility	15
3-1 Introduction	15
3-2 Casting out the Nines	15
3-4 Divisibility by 11	21
Supplementary Unit 4	33
4-1 Introduction	33
4-2 The Necessary Conditions	34
4-3 Discovery of a Property of Graphs	35
Sufficient Conditions	36
Supplementary Unit 5 - Finite Differences	41
5-1 Arithmetic Progressions	42
5-2 More Sequences	42
5-3 Finding Formulas that Fit	43
Supplementary Unit 6 - The Latest Dope On Primes	53

COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 1

SETS

This is a supplementary chapter on sets. Although it is made available with the more adept pupils in mind, the average pupil should be able to follow the development. It should be possible to cover the material in six to eight (50 minute period) lessons, including a test.

The idea of "set" has been used rather extensively in this course, beginning with Chapter 2. The general structure of sets, however, has not been extensively developed. In view of the importance and usefulness of set theory in mathematics, especially in the understanding of algebra, it is thought desirable to include this chapter for possible use and enrichment.

Because of the way in which the text is developed, it is not necessary to have much additional material in this manual.

1-1. Introduction

This section is in the nature of a review. It explains that a set is merely a collection of objects, any objects, and calls the pupils' attention to the fact that the idea of "set" has been used elsewhere in this book.

1-2. Sets, Their Number and Their Subsets

This section begins by explaining further the concept of set. Following this, sets are defined and discussed from the standpoint of their members, or elements, of a set. Notice especially the paragraph headed "Property." Be sure the symbol "{ }" is understood and properly used.

Exercises 1-2-a. Answers

1. (a) Names of girls.
   (b) U.S. Presidents and Generals.
   (c) The odd counting numbers.
   (d) Multiples of 2, 4, 6, 12.

2. (a) Tom is a member of the set whose members are Carl, Jim, Tom, and Robert.
   (b) 6 is a member of the set of all even whole numbers.
   (c) If X is a member of the set whose members are Tom, Carl, Bob, and Jim, then X must be equal to Tom, or equal to Carl, or equal to Bob, or equal to Jim.

3. (a) True (c) False (e) False (Washington, D.C. is not a state.)
   (b) True (d) True

4. (a) {2, 3, 6}
   (b) {violin player, viola player, cello player}
   (c) {0, 1, 2, 3, 4, 5, 6, ··· }
   (d) {Hoover, Roosevelt, Truman, Eisenhower}.

Always feel free to introduce additional problems, especially problems which will interest the pupils.

Subsets

A set R is a subset of a set S if every element of R is an element of S. It is important that the pupils understand this definition and the use of the symbols, "$\subset$" and "$\supset$". The pupils should enjoy the Venn diagrams. Perhaps you will want to introduce more examples of diagrams.

Exercises 1-2-b. Answers

1. (a) If X is a member of the set of all red flowers, then X is a member of the set of all flowers.
   (b) M is contained in N, and N contains M.
   (c) The set of the odd counting numbers is contained in the set of all counting numbers.

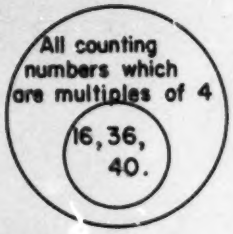
2. {4}, {5}, {6}, {4,5}, {4,6}, {5,6}, {4,5,6}.

3. (a) {12, 20, 32} $\subset$ {0, 1, 2, 3, 4, $\cdots$ }.
   (b) {Lake Superior, Lake Michigan, Lake Huron, Lake Erie, Lake Ontario} $\supset$ {Lake Huron, Lake Michigan}.
   (c) {Hoover, Truman} $\subset$ {Wilson, Harding, Coolidge, Hoover, Roosevelt, Truman, Eisenhower}.

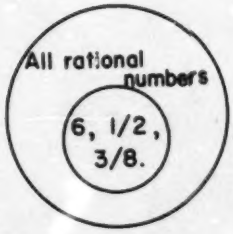
4. (a)

(b)

4. (c)



(d)



5. (a) False

(b) False

(c) True

(d) True

6. Yes

1-3. Operations with Sets

In this section the operations (union and intersection) with sets are introduced. Also the definition of equality (identical sets) is given. In addition the commutative, associative, and distributive properties are discussed. The similarity in the use of these properties with sets and with the rational numbers should be discussed.

Exercises 1-3-a. Answers

1. (a) $M \cup N = \{\text{Red, Blue, Green, Yellow, White}\}$.

   (b) Yes. Commutative property.

2. (a) Yes. $(A \cup B) \subset C$, and $C \subset (A \cup B)$

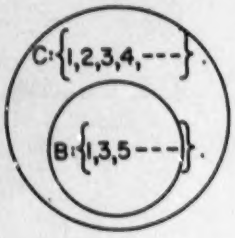
   (b) Yes. A is a subset of C.

   (c) No. A and B have no elements in common.

2. (d) Yes. Commutative property.

(e) No. A and B have no elements in common.

(f)



(g) No. A is not contained in B and B is not contained in A.

3. (a) Yes. Associative property.

(b) Yes. The definition of equality.

4. (a) True. Since X is a subset of C and C is a subset of S, then X is a subset of S and is, therefore, a member of S.

(b) True. C is a subset of S.

(c) False. $C \subset S$, but S is not contained in C.

(d) False. S is not a subset of C.

(e) True, X is a subset of C.

(f) True. C is a subset of S.

(g) Yes, since $X \in C$.

Yes, since $X \in S$.

(h) Yes, since $S \subset C$.

5. (a)

(b) $Y \subset X$. Y is a subset of X.

Exercises 1-3-b. Answers

1. (a) $A \cap B = \{girl, chair\}$.

   (b) $A \cap C = \{chair\}$.

   $C \cap A = \{chair\}$.

   (c) $A \cap (B \cup C) = \{boy, girl, chair\} \cap (\{girl, chair, dog\} \cup \{chair, dog, cat\}) = \{boy, girl, chair\} \cap \{girl, chair, dog, cat\} = \{girl, chair\}$.

   $(A \cap B) \cup (A \cap C) = (\{boy, girl, chair\} \cap \{girl, chair, dog\}) \cup (\{boy, girl, chair\} \cap \{chair, dog\ cat\})$
   $= \{girl, chair\} \cup \{chair\}$
   $= \{girl, chair\}$.

   (d) $A \cap (B \cup C) = \{boy, girl, chair\} \cap (\{girl, chair, dog\} \cap \{chair, dog, cat\})$
   $= \{boy, girl, chair\} \cap \{chair, dog\}$
   $= \{chair\}$.

   $C \cap (A \cap B) = \{chair, dog, cat\} \cap (\{boy, girl, chair\} \cap \{girl, chair, dog\})$
   $= \{chair, dog, cat\} \cap \{girl, chair\}$
   $= \{chair\}$.

2. Yes. Commutative property.

   Yes. $\emptyset$ is a subset of H.

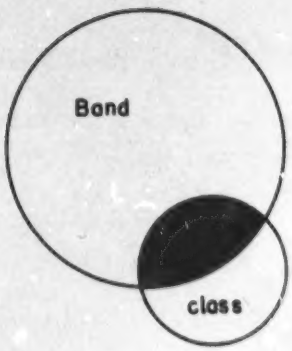
   Identity element.

3. (a) Parallel.

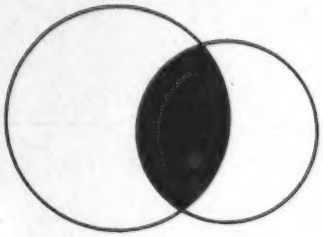
   (b) They intersect.

4. Yes. The commutative, associative and distributive properties apply to both sets.

5.



6.



7. (a) Odd counting numbers.

   (b) $\emptyset$.

8. (a) Yes. A is a subset of B.

   (b) Yes. The intersection of two sets contains only those elements which are common to both sets.

1-4. Order, One-to-One Correspondence, the Number of a Set, and Counting

The first concept we will consider is that of <u>order</u>. As pointed out in the text, <u>ordered sets</u> are very important in many branches of mathematics. The concept is also important in other fields of science, as well as many other areas. For example, ordered pairs are most important in a shoe store.

The second concept is a basic one: That of one-to-one correspondence. The ideas developed here should be thoroughly understood by the pupils.

The parts devoted to the number of a set, and to counting are also basic. These topics should be interesting and should increase the pupils' experience in working with sets.

Exercises 1-4-a. Answers

1.

Set A		Set B
Bill	⟷	Jane
Tom	⟷	Ann
Sam	⟷	Susan

Set A		Set B
Bill	⟷	Susan
Tom	⟷	Ann
Sam	⟷	Jane

Set A		Set B
Bill	⟷	Jane
Tom	⟷	Susan
Susan	⟷	Ann

Set A		Set B
Bill	⟷	Susan
Tom	⟷	Jane
Susan	⟷	Ann

2. 24

3. (a) True. For example: $\{7,8,9\} = \{7,9,8\}$, and also $\{7,8,9\} \longleftrightarrow \{7,9,8\}$.

(b) False. For example: $\{1,2,3\} \neq \{7,8,9\}$, but $\{1,2,3\} \longleftrightarrow \{7,8,9\}$.

(c) True. See example for Part (a).

3. (d) False. See Parts (a) and (c).
   (e) False. See Part (a).

4. (1, x)
   (2, y)
   (3, t)
   (4, a)
   (5, b)
   (6, c)

5. No. One-to-one correspondence between the set of eggs and set of receptacles in the carton.

6. Yes. For every element in X there is a corresponding element in y, and for every element in y, there is a corresponding element in X.

7. No. There is a one-to-one correspondence between the two sets of points.

Exercises 1-4-b. Answers

1. (a) 6. (c) 5. (e) a and d.
   (b) 4. (d) 6.

2. $R \leq S$.

3. M matches R.

4. The same number.

5. C_{12}: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.
   C_7: {1, 2, 3, 4, 5, 6, 7 }.

C_{12}	C_7
1 ↔	1
2 ↔	2
3 ↔	3
4 ↔	4
5 ↔	5
6 ↔	6
7 ↔	7
8 ↔	
9 ↔	
10 ↔	
11 ↔	
12 ↔	

6. A ∪ B = {Bob, Sue, Tom, Joe, cat, dog, chair}.
   C = {1, 2, 3, 4, 5, 6, 7, 8, 9, ··· }.
   A ∪ B = C_7.

7. M ∪ N = {1, 2, 3, 4, 5, 6, 7, 8, 9}.
   C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ··· }.
   M ∪ N = C_9.

COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 2

SPECIAL FIGURES IN PROJECTIVE GEOMETRY

This brief unit is a little introduction to projective geometry. The basic idea is a change in language which replaces "two lines are parallel" by "two lines intersect in an ideal point." It should be emphasized that this is just a change in language and does not mean that two parallel lines intersect in a point "way out there." However, the change is a fruitful one because with this interpretation we can say that "any two lines determine a point." This is the basis of the "principle of duality" in projective geometry which is only hinted at in the text. Roughly speaking it means that if a theorem about a plane configuration composed of lines and points is true, then its "dual" theorem in which the words "line" and "point" are interchanged is also true. In a certain sense, Desargues' theorem is "self-dual."

Desargues' theorem, beside being one of the basic theorems of projective geometry is very interesting in itself and the pupils should get much enjoyment out of drawing various figures showing this configuration. Problems 2 and 3 in Exercises 3 are especially fruitful in this connection.

This unit depends on very little of the content of Junior High School Mathematics, Vol. 1. Pupils would probably enjoy it

more after they have studied Chapter 4, than they would before their introduction to non-metric geometry.

Exercises 1. Answers

1. Here ideal points of one line correspond to ideal points of the other.

2. One-to-one correspondence.

3. The line PQ.

4. The line through P parallel to AB. This line intersects ℓ in its ideal point.

5. This means: There is just one line through P parallel to ℓ.

6. (a) Ideal point.
   (b) To get the point on ℓ_1 corresponding to G', draw a line through G' parallel to the four parallels and cutting ℓ_1 in G. A similar construction gives the other two.

Exercises 2. Answers

1. Figures are called for.
2. CO and QA' intersect in C'; OB and A'P intersect in B; BC and PQ intersect in R. C', B', and R are collinear.

3. One may take any other three points for the first triangle.

One choice would be AQA'. Then the second triangle would have to be BRB'. Then AQ and BR intersect in C; QA' and RB' intersect in C'; A'A and B'B intersect in O. Then C, C', and O are collinear.

4. Actually the same kind of figure would appear but it would be drawn differently.

5. This is really a matter of verification. This configuration is called a "finite geometry" - it consists of 7 points and 7 "lines." (See B. W. Jones "Miniature Geometries" The Mathematics Teacher, vol. 52 (1959) pp. 66-71).

COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 3

REPEATING DECIMALS AND TESTS FOR DIVISIBILITY

3-1. Introduction

This unit is for unusually good students. One of the purposes of this monograph is to encourage the student to be self-reliant The teacher should only give him as much help as is necessary to keep him going and refuse help when it becomes apparent that the student wants the teacher to save him mental effort. The teacher might even want to make application of these principles himself and see how much he can do without the help of this commentary.

3-2. Casting out the Nines

Here the student should see the advantage of multiplying the numbers "algebraically" using the distributive property and commutative property. Probably one of the hardest things to see in this section is that if $9b + r$ is divided by 9, the remainder is the same as when r is divided by 9. Fundamentally the explanation is this: Every number can be divided by 9 yielding a quotient, q, and a remainder r. Thus if N is the number, it can be written $9q + r$ where r is the remainder, that is, is zero or a natural number less than 9. Conversely, if a number is written in the form $9q + r$ where r is a natural number less than 9, then r is the remainder when the number $9q + r$ is divided by 9. Thus what the expansion of 156,782 in powers of ten shows, is

that it can be written in the form $9q + (1+5+6+7+8+2)$. Then the sum in parentheses can be written in the form $9t + 2$, where 2 is the remainder after division by 9. Hence 156,782 is equal to $9(q+t) + 2$ which means that 2 must be the remainder when the sum of the digits is divided by 9. This is also dealt with in the students' material. The student can probably see this better in terms of numbers, although it may be easier for the teacher to see in letters. While the student should eventually get to the use of letters, the important thing is for him to see what is happening. This can probably best be done by working out a number of numerical problems along the lines given in the text.

Thus, in Exercise 2, the student might write $69 + 79 = 7 \times 9 + 6 + 8 \times 9 + 7 = 7 \times 9 + 8 \times 9 + 6 + 7 = (7 + 8) \times 9 + 6 + 7$. He should be required to name what properties of numbers (commutative, associative and distributive properties) he is using. This is a good opportunity to impress these properties on him, but it should not be done to the point of boredom. When he gets to the point that it is clear that he knows what is going on, he should not be required to name the property each time. For a sum of three he could show from first principles or, by combining terms, reduce it to summing two things twice just as $a + b + c = (a + b) + c$.

In Exercise 3, the student would of course write 69 as $9 \times 7 + 6$ and 79 as $11 \times 7 + 2$. In Exercise 4, one would proceed as above except that one would multiply instead of add. The same results would hold for 23 since one could write 79 as $3 \times 23 + 10$ and 69 as $3 \times 23 + 0$. They would hold equally well for any number.

The general principle should be formulated as soon as the student can do so himself. The remainder when the sum (or product) of any two numbers is divided by a given number, is the same as when the sum (or product) of the remainders is divided by a given number.

In Exercise 5 the answer is 1^{20}, which is 1 whether the divisor is 9 or 3. If the divisor is 99, we can write 10^{20} as 100^{10} and see that then the remainder will be 1^{10} which is also 1. The bright student might want to know what would happen if the divisor were 999. Then one would write 10^{20} as $1000^{6} \times 100$ and so the remainder would be 1×100 which is 100. By a similar argument the remainder when 7^{20} is divided by 6 is 1.

The student might want to carry this farther and consider the remainder when 10^{20} is divided by 7. This could be worked out as follows: 10^{20} could be replaced by 3^{20} which is equal to 9^{10}. That remainder would be the same as that for 2^{10} which is $(2^3)^3 \times 2$ or $8^3 \times 2$ which has the remainder $1^3 \times 2$ or 2. A student bright enough to be interested in developing this should have no trouble with manipulating the exponents. He might want to explore what would be the last digit in the huge number 3^{40}. This would be just the remainder when 3^{40} is divided by 10. Since 3^4 has a remainder of 1 when divided by 10, the answer would be 1^{10} or 1. This exercise might also be done by looking at the pattern of last digits in the powers of 3: 3, 9, 27, 81. The last digits form the pattern 3, 9, 7, 1, 3, 9, 7, 1 and so forth. These things could be explored still further.

For Exercise 7 the simplest test for divisibility by 4 is to test whether the number consisting of the last two digits is divisible by 4. For instance, 178524 is divisible by 4 since 24 is. This is because 178524 is equal to 178500 + 24 and any multiple of 100 is divisible by 4. Similarly to test for divisibility by 8 one uses the last three digits. Another test for divisibility by 4 would be to see that if the last digit is 0, 4 or 8 the number is divisible by 4 if the next to the last digit is even. If the last digit is 2 or 6, the number is divisible by 4 if the next to the last digit is odd. The reasons should of course be found.

For Exercise 8, the number written to the base twelve would be a multiple of twelve plus the last digit. Thus the number would be divisible by any divisor of 12 if the last digit is divisible by this divisor. That is, if the number is divisible by 6, its last digit must be divisible by 6. Other numbers in place of 6 would be 2, 3, 4, 12. If the base were 7, since 7 has no factors but itself and 1, the only number that could correctly go in the blank would be 7.

Exercise 9 is rather fundamental but requires some insight on the part of the student. Just as a decimal terminates when the divisor is a divisor of a power of 10, so a "decimal" to be base 7, will terminate when the divisor is a divisor of a power of 7. That is, 1/7 and 1/49 will have terminating "decimals" to the base 7. The decimal expansion of a fraction depends on the number base. But whether or not it is rational is independent of the base. A number that is a prime number when expressed in the decimal notation

is a prime number when expressed to any other base, since the property of being prime is a property of the number and not of the way in which it happens to be written. You may not want to tackle this point at this stage with your students, but you perhaps should be prepared to meet it if it occurs.

The answer to number 10 would be: the remainder when $2 + 2 + 2 + 2 + 2$, or 10, is divided by 7. Hence the remainder is 3. Number 11 is easy and the answers to numbers 12 and 13 are developed in the text after the exercises.

In what follows it is probably wise to deal with the discussion with letters since here the advantage is quite clear and the manipulation is not very complex.

Probably the students will be interested in doing quite a little casting out of the nines in numerical examples. There is an amusing little book by E. T. Bell on "Numerology" which the students might like to read.

In the next set of exercises, the justification of the casting out of nines is that the remainder when the sum (or product) of two numbers is divided by 9 is the same as the remainder when the sum of the digits is divided by 9, as well as the properties described in Exercises 2 and 4 of the previous sets. For instance, suppose a product ab is to be checked. The remainder when ab is divided by 9 is the same as the remainder when the product of their remainders is divided by 9 which is the same as the remainder when the product of the sums of their digits is divided by 9. So

whichever way one does it, he is computing the remainder after division by 9 -- it does not matter anywhere whether a number or the sum of its digits, is divided by 9 as far as the remainder goes. The check of course only checks the remainder after division by 9. The digits for instance could be scrambled without altering the value of their sum and the answer could be completely wrong. But of course this is not very likely to happen.

For Exercise 4, the remainders would be 3 for division by 7, and, since the remainder when 7 is divided by 6 or 3 is 1, the remainder when the number is divided by 6 or 3 would be the same as when the sum: 5 + 3 + 2 + 1 + 4 + 3 is divided by 6 or 3. The sum is equal to 18 which is divisible by both 6 and 3. Hence the given number is divisible by 6 and 3, that is, has a remainder 0 when divided by 6 or 3.

The short-cuts asked for in Exercise 5 are described in the text that follows. As Exercise 4 shows, one would cast out threes, or sixes or twos in the number system to the base 7, since 6, 2, and 3 are divisors of 7 - 1 just as 3 and 9 are divisors of 10 - 1.

For Exercise 7, it may be seen that scrambling the digits does not alter the value of their sum nor the remainder when the numbers are divided by 9. Hence each of the numbers is of the form 9n + r and 9t + r, with the remainders the same. Then if we subtract one from the other, we get 9n + r - 9t - r which is equal to 9(n - t), that is a multiple of 9. Thus the sum of its digits is a multiple of 9. When the sum of the digits given is a multiple of 9, one

cannot be sure whether the missing digit is 9 or 0. Otherwise the trick can be worked by adding the sum given and seeing what number added to this sum will give a multiple of 9. For instance, in the example given, one must add 4 to 14 to get a multiple of 9 and hence 4 is the missing digit.

3-4. Divisibility by 11.

The exercises are solved in the text which follows them. You may have a little trouble with the product of -1 and -1 and you may want to avoid this; this can be done in the example worked out by seeing that 10^3 is $10^2 \times 10$ which would have a remainder of $1 \times (-1)$ or -1. If you wish to avoid negative numbers completely, you could confine yourself to the first test. Or you could show the test for two-digit numbers by noticing that if the second digit is larger than the first one, the remainder after division by 11 is the second digit less the first one (e.g., $79 = 77 + 2$). If the second digit were smaller than the first one, one could add 11 to it and subtract the first digit, giving a correct remainder (e.g., for 73, subtract 7 from 3 + 11 and see that the remainder is 7). This is somewhat laborious, however.

It is important in testing for the remainder after division by 11 by the second method to start at the right hand end of the number. If you are merely testing for divisibility it does not matter at which end you start.

For Exercise 2, page 62, the test for divisibility by 8 would be analogous to that that for 11 in the decimal system.

For Exercise 3, the number 157,892 is equal to 157(1000) + 892 or 157 × 999 + (157 + 892). Hence if d is any divisor of 999, the remainder when 157,892 is divided by d is the same as when 157 + 892 is divided by d. The prime divisors of 999 are 3, 37. In fact $999 = 3^3 \times 37$. This test would work for any divisor of 999, that is, for 3, 9, 27, 37, 3 × 37, 9 × 37 as well as, of course, for 999 itself.

The students might also be interested in writing 157,892 as equal to 157(1001) + 892 - 157. Then if d is any divisor of 1001, the remainder when 157,892 is divided by d is the same as when 892 - 157 is divided by d. The prime divisors of 1001 are 7, 11, and 13. This would give a test for divisibility by 7 and 13 as well as 11. This has some connection with the fact that the decimal equivalents of 1/7 and 1/13 have six digits in the repeating portion. This can be developed further.

For Exercise 4, consider a number $534,623_{seven}$. Then in the numeral system to base seven, 534,623 is equal to $534_{seven} \times 666_{seven} + 534_{seven} + 623_{seven}$. Hence we would have to find the divisors of 666_{seven}. Now $666_{seven} = 6_{seven} \times 111_{seven}$. To see if 111_{seven} has factors it is probably easiest to convert to the base ten. So $111_{seven} = 49 + 7 + 1 = 57$ which has 3 and 19 as its factors. Actually we could have tested 111_{seven} for divisibility by 3 by adding the digits since this works for 3 in the system to the base seven as well as to the base ten. So in the number system to the base seven, this kind of test would work for 19, that is 25_{seven} as well as for 2, 3, 6 which are written

the same way in both systems.

Similarly for the number system to the base twelve, we would seek the divisors of eee_{twelve} which is $e_{twelve} \times 111_{twelve}$ and e stands for "eleven." We cannot say that 111_{twelve} is divisible by 3 since 3 is not a divisor of one less than the base. But we have to convert it to the decimal notation. It is 144 + 12 + 1 = 157, which is a prime number. Hence a number system to the base twelve, this kind of test would work for 3 and 157 in the decimal system, that is 3_{twelve} and 111_{twelve}.

The connection between repeating decimals and tests for divisibility should begin to emerge for the student here. Suppose m is some number which is divisible by neither 2 nor 5. Then some power of 10 is 1 more than a multiple of m. This can be seen as follows. We know that the remainders when the powers of 10 are divided by m are the numbers from 1 to m - 1 inclusive. Since there are infinitely many powers of 10, two of the powers must have the same remainder, that is $10^a - 10^b$ will be divisible by m for some natural numbers a and b. Suppose b is smaller than a, then the difference may be written

$$10^b(10^{a-b} - 1).$$

Since m divides this product and has no factors except 1 in common with 10^b it must divide $10^{a-b} - 1$, which is what we wanted to show. Thus we have shown that some power of 10 has a remainder of 1 when divided by m. Call k the smallest such power greater than zero.

From this we can conclude two things. First, the remainders when we compute the decimal expansion of 1/m will be just the remainders when the powers of 10 are divided by m. As soon as we get a remainder 1, the decimal begins to repeat and not before. So our number k is the number of digits in the repeating portion of the decimal.

Second, we may write any number in the form

$$a + b \times 10^k + c \times 10^{2k} + d \times 10^{3k} + \cdots$$

Since the remainder when 10^k is divided by m is 1, the remainder when our number is divided by m is the same as the remainder when

$$a + b + c + d + \cdots$$

is divided by m. We must notice, of course, that a, b, c, d are not digits in general but they are natural numbers less than 10^k.

How far the student can progress here, remains to be seen. Certainly he should not be pushed.

For casting out the elevens, instead of adding the digits, we would take the units digit, minus the tens digit, plus the hundreds digit and so forth. An example would be this:

$$\begin{array}{rr} 637 & 7 - 3 + 6 = 10 \\ \times\ 152 & 2 - 5 + 1 = -2 \\ \hline \text{product: } 96,824 & \text{product: } -20 \end{array}$$

$$\begin{array}{rr} 4 - 2 + 8 - 6 + 9 = 13 & 0 + 2 = 2 \\ 3 - 1 = 2 & \end{array}$$

We come out with 2 in each case. This test is a little less likely to check when the answer is wrong but it is much more difficult to apply.

For the corresponding trick mentioned in Exercise 7, one cannot scramble the digits indiscriminately. The simplest direction to give would be to reverse the order of the digits and add the two numbers if the number of digits is even but subtract the two numbers when the number of digits is odd. This can be seen as follows: Suppose there are three digits a, b, c. Then the remainder for the given number would be that for a - b + c and for its reverse c - b + a. Then the difference would have zero remainder. If the number has four digits a, b, c, d, the sum for the given number would be - a + b - c + d and for its reversal would be - d + c - b + a. Then the sum would have a zero remainder. Then it would not be sufficient to know the sum of all but one of the digits in the answer. You would have to ask for all the digits but the last, or something like that. This trick is much more complicated. Its only advantage is that it does not have the case of failure that that for nine has.

Exercises 2. Answers

1. (a) No answer required. Examples follow:

Sum of digits 7247 is 20 — Not divisible by 9.

$\frac{20}{5}$ = 22 remainder 2

Sum of digits 427986 is 36 — Divisible by 9.

$\frac{36}{9}$ = 4 remainder 0

Sum of digits 9499794 is 51 — Not divisible by 9.

$\frac{51}{9}$ = 5 remainder 6

Sum of digits 187434 is 27 — Divisible by 9.

$\frac{27}{9}$ = 3 remainder 0.

(b) No answer required. Example follows:

$\frac{7247}{9}$ = 805 remainder 2.

Sum of digits is 20; $\frac{20}{9}$ = 2, remainder 2.

(c) The remainder when a number is divided by 9 is the same as the remainder when the sum of the digits of that number is divided by 9.

(d) No answer required. Examples follow:

$\frac{407}{9}$ = 45 remainder 2; $\frac{11}{9}$ = 1 remainder 2

$\frac{877}{9}$ = 97 remainder 4; $\frac{22}{9}$ = 2 remainder 4

2. (a) The uniqueness property of addition. (Refer to page 54 of the students' text.)

(b) Yes

3. (a) Yes

(b) Same reason as number 2.

4. (a) Yes

(b) Yes

(c) Yes

(d) Yes

5. (a) $\frac{(9 + 1)^{20}}{9}$ has a remainder of 1^{20} or 1.

(b) $\frac{(3 \times 3 + 1)^{20}}{3}$ has a remainder of 1^{20} or 1.

(c) $10^{20} = 100^{10} = (99 + 1)^{10}$ has a remainder of 1^{10} or 1.

6. $\frac{(6 + 1)^{20}}{6}$ has a remainder of 1.

7. (a) divisibility by 4: If the number formed by the last two digits is divisible by 4, then the number is divisible by 4.

   (b) divisibility by 8: If the number formed by the last three digits is divisible by 8, then the number is divisible by 8.

   (c) divisibility by 25: If the number formed by the last two digits is divisible by 25 (e.g., 00, 25, 50, 75), then the number is divisible by 25.

8. 2, 3, 4, 6, 12 (0 is divisible by 12).

9. (a) multiples of powers of 7. (This includes negative powers of 7, e.g., 13/49 in the decimal system is .16 in the system to the base 7.)

   (b) $(.12541\cdots)_7$

10. remainder 3.

11. (a) $\frac{(9+1)^{20}-1}{9}$ $1 - 1 = 0$ (Refer to Exercise 5.)

    (b) $\frac{(6+1)^{108}-1}{6}$ $1 - 1 = 0$ (as above)

12. Cast 9's from the sum as you go along. (See page 56 in students' text.)

Exercises 3-3. Answers

1.
927	sum of digits:	18	sum of digits: 9	sum of digits: 9
865	sum of digits:	19	sum of digits: 10	sum of digits: 1
4635				Product: 9
5562				
7416				
801855	sum of digits:	27	sum of digits: 9.	

2. Answered in the text.

3. 810855.

4. 3, 0, 0.

5. Answered in the text.

6. Casting out the sixes.

7. 7543
   5437
   2106 sum of digits = 9

   2 + 1 + 0 = 3, other digit is 6,
   or 1 + 0 + 6 = 7, other digit is 2,
   or 2 + 1 + 6 = 9, other digit is 0 or 9.

Exercises 3-4-b. Answers

1. (1) a. $758 = 7 \times 10^2 + 58$ which has the remainder $7 + 3 = 10$ when divided by 11 since 100 has the remainder 1.

   b. $758 = 7 \times 10^2 + 5 \times 10 + 8$ which has the remainder $8 - 5 + 7 = 10$.

   (2) a. $7246 = 72 \times 10^2 + 46$ which has the same remainder as has $72 + 46$, that is, $6 + 2$ or 8.

   b. $7246 = 7 \times 10^3 + 2 \times 10^2 + 4 \times 10 + 6$ which has the same remainder as has $6 - 4 + 2 - 7 = -3$, that is, 8.

   (3) a. $81675 = 8 \times 10^4 + 16 \times 10^2 + 75$ which has the same remainder as has $8 + 5 + 9 = 22$. Hence the remainder is 0.

   b. $81675 = 8 \times 10^4 + 1 \times 10^3 + 6 \times 10^2 + 7 \times 10 + 5$ which has the same remainder as has $5 - 7 + 6 = 1 + 8 = 11$. Hence the remainder is zero.

2. Since 8 is 1 more than 7, a number to the base 7 can be tested for divisibility by 8 in the same way that we can test for

divisibility by 11 in the decimal system. For instance, consider $(5326)_7$.

Using the first method we have $(5326)_7 = (53)_7 \times (10^2)_7 + (26)_7$. Since 8 is $(11)_7$, the remainder when $(10^2)_7$ is divided by 8 is 1. Thus the remainder when the given number is divided by 8 is the same as when $(53)_7 + (26)_7$ is divided by $(11)_7$. But $(53)_7 - (44)_7 = 6$ and $(26)_7 - (22)_7 = 4$ and hence the remainder is the same as when $6 + 4$ is divided by 8, that is, 2.

Using the second method we have $(5326) = 5 \times (10^3)_7 + 3 \times (10^2)_7 + 2 \times (10)_7 + 6$. When this is divided by 8 the remainder is the same as that for $6 - 2 + 3 - 5 = 2$.

3. (See page 21 of the teachers' commentary.)
3, 9, 27, 37, 3×37, 9×37, 999.

4. (See page 21 of the teachers' commentary.)

In the numeral system to the base seven, we can test for division by grouping in triples all the divisors of $7^3 - 1 = 342$. These divisors are:

2, 3, 6, 9, 18, 19, 38, 57, 114, 171, 342.

In the number system to the base twelve, we would have the divisors of $12^3 - 1 = 1727$. These factors are 11, 157, 1727.

5. (a) Yes. The remainders when the powers of 10 are divided by 11 are $-1, 1, -1, 1, \cdots$ which has a period of 2 since 11 is a divisor of $10^2 - 1$.

(b) The divisors 3 and 9 listed in the answers to number 3 are divisors of $10^1 - 1$ as well as of $10^3 - 1$. All the others have three digits in the decimal equivalent of their reciprocals and for these, grouping the digits in threes gives a divisibility test.

See also teachers' commentary.

	3	7	9	11	13	17	19	21	37	101	41
1	1	1	1	1	1	1	1	1	1	1	1
10^1	1	3	1	10	10	10	10	10	10	10	10
10^2	1	2	1	1	9	15	5	16	26	100	18
10^3	1	6	1	10	12	14	12	13	1	91	16
10^4	1	4	1	1	3	4	6	4	10	1	37
10^5	1	5	1	10	4	6	3	19	26	10	1
10^6	1	1	1	1	1	9	11	1	1	100	10
10^7	1	3	1	10	10	5	15	10	10	91	18
10^8	1	2	1	1	9	16	17	16	26	1	16
10^9	1	6	1	10	12	7	18	13	1	10	37
10^{10}	1	4	1	1	3	2	9	4	10	100	1
10^{11}	1	5	1	10	4	3	14	19	26	91	10
10^{12}	1	1	1	1	1	13	7	1	1	1	18
10^{13}	1	3	1	10	10	11	13	10	10	10	16
10^{14}	1	2	1	1	9	8	16	16	26	100	37
10^{15}	1	6	1	10	12	12	8	13	1	91	1
10^{16}	1	4	1	1	3	1	4	4	10	1	10

The number of digits in the cycle in the repeating decimal is the same as the number of digits in the cycle of remainders.

Since, for example, there are five remainders in the column headed by 41, one can test for divisibility by 41 by grouping the digits in fives.

For the answers to questions on pages 65 and 66 look on pages 23 and 24 in the teachers' manual.

COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 4

GRAPHS: OPEN AND CLOSED PATHS

4-1. Introduction

The problem of the bridges of Königsberg (called Kaliningrad after the Russians took over in World War II) is one of the first problems ever solved in the field of topology. This is the branch of mathematics which deals with the properties of geometrical figures which are left unchanged by continuous 1-to-1 transformations. You may think of the geometry on a rubber sheet. What properties of the geometrical figures are not changed when you stretch or shrink the sheet in any way whatsoever, so long as you don't tear anything or glue parts together?

The object of this chapter is to show the children that there are interesting properties of geometric figures which do not involve any process of measurement. Section 3 is probably too difficult for any but the brightest pupils, and might be assigned to individuals as outside reading for extra credit. The other parts are for good, but not exceptional students, and may be suitable for homogeneously grouped classes, as well as for independent work for a few good students.

The problem of the existence of Hamiltonian paths, discussed in Section 4, is an excellent example of a question which any child can understand, but which has never been answered. You can

use this to show the class that mathematics is a growing subject and that there is always more to do in mathematics.

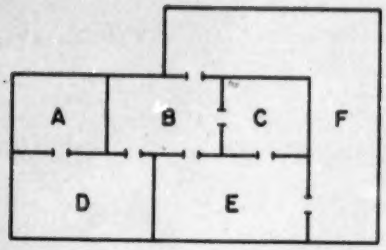
4-2. The Necessary Conditions

In Section 1 we developed the necessary conditions that there be, in a graph, a path which goes over every bridge just once. (Incidentally, in the technical literature of graph theory the term "edge" is used instead of our term "bridge.") The main results are stated in Theorem 1 of Section 2.

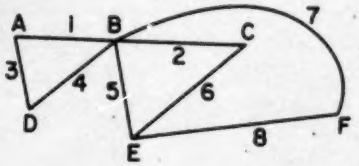
At this level of maturity most children have difficulty with abstract general proofs. They do understand and appreciate "proofs" given in terms of concrete cases, treated by reasoning which they recognize as generally valid. The children will be satisfied with the discussion of Section 1, although the mature student would want a proof of Theorem 1 stated in general terms.

In Figure 1, the vertex A is connected to 5 bridges, while each of the other three vertices is connected to 3 bridges. All 4 vertices are odd (see Section 2). Therefore there is neither an open nor a closed path in this graph which goes over each bridge just once.

You may give the children another concrete situation which leads to the same problem. Here is the floor plan of a house showing 6 rooms and 8 doorways. ... there a path which goes through



each doorway exactly once? We can make a graph



which shows the rooms and their connections. In this graph the vertices B and E are odd and the others are even. The path

B1A3D4B5E8F7B2C6E

goes through each doorway just once. If we insert a doorway connecting room C with room F, then the vertices C and F will also be odd. In that case, there is no path of the desired kind.

Exercises 1 to 4 are preparation for Section 3.

4-3. Discovery of a Property of Graphs

In the exercises of Section 2, the pupil is led to discover (in Exercise 2-2) empirically that the number of odd vertices in any graph is even. The rest of these exercises lead the youngster to discover a proof of this fact.

In Exercises 2-2 and 2-3 you are not supposed to check all the graphs the pupils make for themselves. The important thing is that they discover certain generalizations for themselves. If you teach the chapter to the whole class, you may have several pupils draw graphs on the blackboard and have the whole class check the classification of the vertices as odd or even.

Section 4. Sufficient Conditions

The only parts of Section 3 which might be suitable for more than a few students are the discussions of the distinction between necessary and sufficient conditions and the remarks on the importance of stating exactly what we mean.

Most pupils will overlook the fact that our definition of the term "graph" does not require that a graph be connected. It is instructive for them to learn, by bitter experience, not to jump to conclusions. At first they will probably consider Figure 8 a dirty trick!

On page 83, if Q_8 is not the same as Q_2, then the path is open. The point Q_8 is then an endpoint of the path. Then the path contains an odd number of bridges connected to Q_8. But, by assumption, the total number of bridges connected to Q_8 is even. Then the path cannot contain all of the bridges through Q_8 in the graph.

The proofs of Theorems 2 and 3 are examples of <u>reductio ad absurdum</u>.

Exercises 4-1. Answers

1.

	odd vertices	even vertices	number of odd	number of even
I	A, B, E, G	C, D, F	4	3
II		all	0	12
III	A, C	all others	2	6
IV	A, E	all others	2	3

2. (a) I, III, IV.

   (b) I, II.

3. In II, the path

   A 1 B 2 C 3 D 4 E 5 F 6 G 7 H 8 I 9 J 10 K 11 L 18 -

   B 13 D 14 F 15 H 16 J 17 L 12 A.

   In III, the path

   A 1 B 6 H 10 G 5 B 4 F 8 E 3 B 2 C 11 D 14 G 9 F 13 -

   D 12 E 7 C.

   In IV, the path

   A 1 B 9 C 2 A 7 D 3 B 5 E 6 C 4 D 8 E.

   In each of these there are also many other paths.

4. I. B 2 C 6 D 3 B 4 E 8 F 5 B

   II. B 1 A 12 L 11 K 10 J 9 I 8 H 7 G 6 F 5 E 4 D 3 C 2

   B 13 D 14 F 15 - H 16 J 17 L 18 B

   III. B 2 E 7 C 11 D 12 E 8 F 13 D 14 G 10 H 6 B 5 G 9 F 4 B.

   IV. B 9 C 2 A 7 D 4 C 6 E 8 D 3 B

5. A 1 B 2 C 5 D 4 A 3 C and A 4 D 5 C 3 A 1 B 2 C

Exercises 4-2. Answers

1. See above table.

2. The number of odd vertices in any graph is even.

3.

degree	Number of vertices	I	II	III	IV
1		2	0	1	0
2		2	6	1	0
3		1	0	1	2
4		1	6	4	3
5		1	0	0	0
6		0	0	6	0

4. $V = V_1 + V_2 + V_3 + \cdots$.

6. Two, one for each end of the bridge. The number of pairs is $2 \cdot B$.

7. Each vertex of degree k occurs in k pairs. The total number of pairs in which a vertex of degree k occurs is $k \cdot \frac{1}{2}$.

8. $V_1 + (2 \cdot V_2) + (3 \cdot V_3) + \cdots$

9. $V_1 + V_3 + V_5 + \cdots$

10. $(2 \cdot B) - U = (V_1 + (2 \cdot V_2) + (3 \cdot V_3) + \cdots)$

$- (V_1 + V_3 + V_5 + \cdots) = (2 \cdot V_2) + (2 \cdot V_3) + (4 \cdot V_4)$

$+ (4 \cdot V_5) + \cdots$.

11. By the formula, we have

$$U = (2 \cdot B) - \{(2 \cdot V_2) + (2 \cdot V_3) + \cdots\}$$
$$= 2 \cdot \{B - [V_2 + V_3 + (2 \cdot V_4) + (2 \cdot V_5) + \cdots]\},$$

which is even.

BRAINBUSTER. Let A and B be the two odd vertices. Consider the longest path from A to B which does not go over any bridge more than once. Color this path blue. Continue as in the proof of Theorem 2, Section 4. None of the graphs in Figure 11 has a Hamiltonian path.

COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 5

FINITE DIFFERENCES

A basic method of finding polynomial formulas to fit tables of values is that called "finite differences." Here the student is given a very brief introduction to this theory. The teacher is referred to two books on the subject:

1. C. H. Richardson "An Introduction to the Calculus of Finite Differences" Van Nostrand (1954) 142 pages.

2. A book on the same subject by F. H. Miller, published by Henry Holt and Company to appear the latter part of 1959 or early in 1960.

These two books are too advanced for a seventh or eighth grader - in fact for most high school students - but they should be intelligible to anyone who has had a first course in calculus. This does not mean that calculus forms the basis for finite differences, but merely that the analogies between the formulas in the two subjects are very striking.

Though an attempt is made to keep the algebra used in this unit to a minimum some is certainly needed; for instance the expansion of $(n + 1)^3$ is needed. Of course in this unit as in the other supplementary ones, the student should do a large body of experimentation himself and be given the opportunity to make guesses and verify them.

5-1. Arithmetic Progressions

Here the arithmetic progression is taken up from the standpoint of finite differences and the relationship with the triangular numbers is exploited. The student should see both the geometrical and algebraic arguments and how they are related. Though in the exercises a basis is laid for the generalformula for the sum of an arithmetic progression, more stress is laid on finding the sum than either in getting or using the formula. The most important idea of the section is stated formally in the theorem in Section 3. In all the exercises and examples except problem 12 the entries in the table are increasing. Possibly more examples should be given in which the differences are negative.

In Table II the entries might be extended one to the left with a zero in row 1 and a 1 in row 2. Then the entries in the second row of Table II would be just the entries in the first row of Table I.

The pupil might be interested to notice that the triangular numbers are those on the second slanting line of the Pascal triangle. The pyramidal numbers defined in problem 6 of Exercises 3, are those in the third slanting line of the Pascal triangle.

5-2. More Sequences

There are two methods for finding the formula for the sum of the first n squares. Both rest on the fundamental fact that in any table, the n-th term minus the first term, is the sum of the first differences to the left of the n-th term. This section

works downward in the table and finds the formula for the sum of squares by starting with the cubes. (The second method, that in Section 3, works upward in the table and establishes the formula by guessing what form it should take and determining the appropriate coefficients.)

By way of introduction here, the formula for the sum of the first n integers is gotten from Table III, the table of squares. This is a more complex way of doing it but it does illustrate the method which will be used in the latter part of the section.

Gradually there should emerge the idea that a linear formula fits a table of numbers in an arithmetic progression, a quadratic formula if the first differences are in an arithmetic progression, a cubic formula if the second differences are in an arithmetic progression, etc.

Also in the exercises of the sections the idea is developed that a linear formula can be found to fit any two numbers since any two numbers can be used to start an arithmetic progression; a quadratic formula can be used to fit any three numbers since any three numbers can be used for the first three entries in a table whose first differences form an arithmetic progression, etc.

5-3. Finding Formulas that Fit

Here the method is used which guesses at a formula for the table and then determines the constants by means of the first and second differences.

The product of any three consecutive integers must be divisible by 6 since at least one is even and at least (in fact, only) one is divisible by 3. For four consecutive integers, one must be divisible by 2, one by 4 and one by 3; in other words the product must be divisible by 24. A bright student might like to see what happens for five consecutive integers.

Triangular, square, pentagonal, etc. numbers are called "figurate numbers" or "polygonal numbers." The general formula for the n-th k-gonal number is

$$(k - 2) \times \frac{n^2 - n}{2} + n.$$

That is, when k is 3 we have triangular numbers, $k = 4$ gives square numbers, $k = 5$ gives pentagonal numbers, etc. Notice that they are all quadratic expressions and hence the first differences are in an arithmetic progression.

Of course one may have numbers associated with pyramids whose bases are squares, pentagons, etc.

Exercises 5-1. Answers

1. All these sequences would be of the form $b + n$ for some b. The sum $(b + 1) + (b + 2) + \cdots + (b + n) = nb + (1 + 2 + \cdots + n)$ $= nb + \frac{1}{2}n(n + 1) = \frac{1}{2}n(2b + 1 + n)$. This could also be gotten directly by using the methods above.

2. The sum would be: $(b + 2) + (b + 4) + \cdots + (b + 2n) =$ $n(n + b + 1)$.

3. The next three entries are: 21, 23, 25. The differences are all the coefficient of n in $2n + 7$.

4. For $3n + 7$ the first five values are: 10, 13, 16, 19, 22 and the differences are 3, the coefficient of n; for $2n + 6$ the first five values are 8, 10, 12, 14, 16 and the differences are 2, the coefficient of n.

5. The differences are 5. In fact, if the form is $an + b$, the next term would be $a(n + 1) + b$ and the differences are all a. This is leading up to the theorem in Section 3.

6. Using the trick used at the end of Section 1, we write

$$\begin{array}{ccccccc} 1 & 3 & 5 & \cdots & 2n-5 & 2n-3 & 2n-1 \\ 2n-1 & 2n-3 & 2n-5 & \cdots & 5 & 3 & 1 \end{array}$$

and the sum is $\frac{1}{2}(2n)n = n^2$. See also the next section.

7. Replacing n by $n - 1$ in the formula for the n-th triangular number, we get $\frac{1}{2}(n - 1)(n - 1 + 1) = \frac{1}{2}n(n - 1)$.

8. We can use the trick of problem 6 to get

$$\frac{1}{2}(1 + 1 + nd)(n + 1) = \frac{1}{2}(2 + nd)(n + 1)$$

or it can be done by writing the sum:

$$(n + 1) + d(0 + 1 + 2 + \cdots + n).$$

9. In the above replace n by $n - 1$ to get

$$\frac{1}{2}n[2 + (n - 1)d].$$

10. Add $(b - 1)n$ to the previous answer to get

$$\frac{1}{2}n[2b - 2 + 2 + (n - 1)d] = \frac{1}{2}n[2b + (n - 1)d].$$

11. 7 12 17 22 27

5 5 5 5

We add 5 each time.

12. 7 5 3 1 -1 ··· Here we continue to subtract 2.

13. To construct such a table, compute the first difference from the given two numbers and then add this difference to each number of the table to get the next.

Exercises 5-2. Answers

1. To get a square of $(n + 1)$ dots on a side from one with n dots on a side, we can add two edges of n dots each and a 1 in the corner, which accounts for the $2n + 1$.

2. The methods of the section may be used or the sum can be made to depend on that already found as follows:

$$2^2 + 4^2 + 6^2 + \cdots + (2n)^2$$
$$= 4 \times 1^2 + 4 \times 2^2 + 4 \times 3^2 + \cdots + 4 \times n^2$$
$$= 4 \times \frac{2n^3 + 3n^2 + n}{6} = \frac{2}{3} \times (2n^3 + 3n^2 + n).$$

3. Again the methods of the section may be used or the following:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$$
$$= (2 \times 1 - 1)^2 + (2 \times 2 - 1)^2 + \cdots (2n - 1)^2$$
$$= (4 \times 1^2 - 4 \times 1 + 1) + (4 \times 2^2 - 4 \times 2 + 1) + \cdots$$
$$(4n^2 - 4n + 1) = 4s - 4 \times \tfrac{1}{2}(n^2 + n) + n$$
$$= (2/3)(2n^3 + 3n^2 + n) - 2n^2 - 2n + n = \tfrac{1}{3} \times (4n^3 - n).$$

4.

4	7	12	19	28	39
	3	5	7	9	11
		2	2	2	2

where we first write the third row, then the second and then the first.

5.

10	5	11	28	56
	-5	6	17	28
		11	11	11

6. Starting with any three numbers we could construct such a table if we get the first two first differences and the first second difference and then compute the table from the bottom up.

7. Here we would use the table of fourth powers and first compute the difference between the $(n + 1)$st and the n-th fourth powers, getting $4n^3 + 6n^2 + 4n + 1$ as the n-th term in the set of first differences. Taking the sum, as we did above, we have

$$4 \times (\text{sum of cubes}) + 6 \times (\text{sum of squares}) + 4 \times (\text{sum of integers}) + n = (n + 1)^4 - 1.$$

Let C stand for the sum of the first n cubes and use the formula above the sum of squares to get

$$4C + 6 \times \tfrac{1}{6}(2n^3 + 3n^2 + n) + 2n(n + 1) + n = n^4 + 4n^3 + 6n^2 + 4n.$$

Solving for C, we get

$$C = \frac{1}{4}(n^4 + 2n^3 + n^2) = [\frac{1}{2}n(n + 1)]^2.$$

Notice that the formula for C is the square of the formula for the sum of the first n integers.

Exercises 5-3-a. Answers

1. (a) 6 12 20 30 42

   (b) 0 1 4 10 20

   Except for the entry 0, these numbers are one-sixth of those in the table for $n(n + 1)(n + 2)$. The reason is that $n^3 - n(n - 1)n(n + 1)$ which is the product of three consecutive integers whose middle one is n. The result seems more interesting if one notices that this means that if a number is subtracted from its cube, the result is the product of three consecutive numbers of which the middle one is the given number. (Here "number" means integer.)

   (c) 2 10 30 68 130

2. Since $n^4 - n^2$ is of the fourth degree, the third differences should be in an arithmetic progression. To check this write

0	12	72	240	600	1260
	12	60	168	360	660
		48	108	192	300
			60	84	108
				24	24

Exercises 5-3-b. Answers

1. (a) The first differences are 5 and hence the numbers are in arithmetic progression of the form 5n b. But

$5 + b = 2$ which implies $b = -3$ and the formula is $5n - 3$.

(b) Here the first and second differences are

14 24 34 44

10 10 10

Hence the first differences are of the form $10n + b$ and since the first one is 14, $b = 4$. So the n-th first difference is $10n + 4$.

The formula to fit the table should be like: $an^2 + bn + c$ and, as in the section we have

$$10n + 4 = 2an + (a + b).$$

Thus $a = 5$, $b = -1$ and $5 = 5 \times 1^2 - 1 \times 1 + c$, implies $c = 1$. So the formula is $5n^2 - n + 1$.

(c) Here the first and second differences are

12 18 24 30

6 6 6

Here the formula for the first differences is $6n + 6$ and

$$6n + 6 = 2an + (a + b)$$

implies $a = 3$, $b = 3$. Then $8 = 3 \times 1^2 + 3 \times 1 + c$ implies $c = 2$ and the formula is $3n^2 + 3n + 2$.

2 and 3. The first, second and third differences are

8 20 38 62 92

12 18 24 30

6 6 6

Notice that the first differences here are the numbers in table

1(c). (If this were not the case, we would compute the formula for the first differences as we did above.) Hence the formula for the n-th first difference is $3n^2 + 3n + 2$.

Since the second differences form an arithmetic progression we guess that the formula for the numbers of the table must be $an^3 + bn^2 + cn + d$ for proper choices of a, b, c, and d. The difference between this and the next term in the first row is

$$a(n+1)^3 + b(n+1)^2 - c(n+1) + d - (an^3 + bn^2 + cn + d)$$
$$= a(3n^2 + 3n + 1) + b(2n + 1) + c$$
$$= 3an^2 + (3a + 2b)n + (a + b + c).$$

Since this must be equal to $3n^2 + 3n + 2$, we must have $3a = 3$, $3a + 2b = 3$, $a + b + c = 2$ and hence $a = 1$, $b = 0$, $c = 1$ which reduces the formula to

$$n^3 + n + d.$$

Take the first term to determine d: $2 = 1^3 + 1 + d$ which gives $d = 0$ and the formula is

$$n^3 + n.$$

(See problem 1(c) in Exercises 3-a.)

4. To get the number 15, we add four sides of 3 dots each and subtract 3 duplicates to get $6 + 9 = 15$. The next time we add four sides of four dots each and subtract 3 duplicates (this is just 4 more than we added the previous time). This gives $15 + 13 = 28$, etc. Formula: $2n^2 - n$.

5. The first two lines of the table for the sums of squares are

1	5	14	30	55	
	4	9	16	25	$(n+1)^2$

Notice that the n-th term in the second row is $(n + 1)^2$ instead of n^2 since we started with the second square. Since the second differences in this table are in an arithmetic progression, we know that the formula should be of the form

$$an^3 + bn^2 + cn + d.$$

Then, as in the solution of problem 3, we have

$$(n + 1)^2 = n^2 + 2n + 1$$
$$= 3an^2 + (3a + 2b)n + (a + b + c)$$

which gives: $a = 1/3$, $2 = 1 + 2b$ or $b = \frac{1}{2}$, $1 = 1/3 + \frac{1}{2} + c$, or $c = 1/6$ and the formula is

$$\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n + d.$$

Since when $n = 1$ this must have the value 1, d must be zero and our final formula is, as before,

$$\tfrac{1}{6}(2n^3 + 3n^2 + n).$$

6. The n-th pyramidal number is the sum of the first n triangular numbers. By the same method as that above, we can get the formula $\frac{1}{6}(n^3 - n)$ which is the formula for the table in problem 1(b) in Exercises 3-a.

7. A polynomial of the fourth degree

8. This is merely a matter of checking numerical values. Anyone interested in learning more about polygonal numbers should look in Volume 2 of "The History of the Theory of Numbers" by L. E. Dickson.

9. In each case the first differences repeat the table except for the initial entry in (b). Suppose the entries were

$$a \quad b \quad c \quad d \quad \cdots$$

If the first differences gave the same table we would have

$$a = b - a, \quad b = c - b, \quad \cdots$$

and since the first equation implies $2a = b$, and similarly for the others, the entries would have to be

$$a \quad 2a \quad 4a \quad 8a \quad \cdots$$

in geometric progression.

If the second first difference were the first term of the table, we would have $c - b = a$, that is, $c = b + a$. So if this property were to continue throughout the table, each entry would have to be the sum of the previous two. The student can carry this further if he is interested.

10. An arithmetic progression has a formula $an + b$ associated with it.

11 and 12. See the teachers' manual.

COMMENTARY FOR TEACHERS

SUPPLEMENTARY UNIT 6

THE LATEST DOPE ON PRIMES

The main purpose of this unit is to give the children some idea of mathematics as a living, growing science. We try to drive this home by relating Unit IV to a paper published in October, 1958, thus emphasizing that even today mathematical discoveries which they can understand are being made. A minor goal is to show how mathematical theory is used in work with modern computers. You may stress that the computer is a giant idiot slave which does exactly what you tell it to do, neither more nor less, and that someone with brains still must give the instructions to the computer.

In computing the number of seconds in a year, teach the children that we don't want to know the exact number, but only the order of magnitude. We therefore round off the numbers freely in order to simplify our work. Ask, "How many seconds are there in a minute? How many minutes are there in an hour? How many seconds are there, then, in an hour?" Don't carry out the calculation, just indicate it:

60×60 seconds in an hour.

"How many hours are there in a day, and how many days in a year?" Show the total number of seconds in a year:

$60 \times 60 \times 24 \times 365$ seconds in a year.

Even here we are approximating, since a year is about $365\frac{1}{4}$ days.

We can round off thus:

$60 \times 60 = 3{,}600 = 4{,}000$ approximately,

$4{,}000 \times 24 = 4{,}000 \times 25 = 100{,}000$ approximately, and

$100{,}000 \times 365 = 100{,}000 \times 400 = 40{,}000{,}000$ seconds in a year, or $4 \cdot 10^7$, approximately. The actual number is 31,536,000, ignoring the error of $\frac{1}{4}$ of a day. The important thing is that this is of the order of 10^7, or an 8 digit number in decimal notation.

The number of years in 10^{97} seconds is

$$\frac{10^{97}}{4 \cdot 10^7} = \frac{10 \cdot 10^{96}}{4 \cdot 10^7} = (2.5) \cdot 10^{89},$$

which is a 90 digit number. Have the children look up the geologists' estimate of the age of the earth. See, for example, Gamow's book, the Biography of the Earth.

You may review the symbols for inequality:

$\neq$ is unequal to,

$<$ is less than,

$>$ is greater than,

$\leq$ is equal to or less than,

$\geq$ is equal to or greater than.

It is helpful, in keeping straight the meanings of " $<$ " and " $>$ ", to remind the children that the smaller end of the symbol points to the smaller number.

At the rate of one division per .001 seconds, it would take about $3 \cdot 10^{39}$ years to perform 10^{50} divisions.

Discussion of Exercises 1 and 2

Exercise 1.

m	1	2	3	4	5	6	7	8	9	10	11	12
n	1	3	7	15	31	63	127	255	511	1023	2047	4095

m	13	14	15	16	17	18	19	20
n	8191	16383	327677	65535	131071	262143	524287	1048575

2. a. 3 divides 3, 15, 63, 255, etc.

b. 7 divides 63, 511, 4095, etc.

c. 31 divides 31, 1023, 32767, and 1048575.

d. If m is divisible by b then n is divisible by $2^b - 1$.

Note the difference between

$$3^{(2^4)} = 3^{16},$$

$$(3^2)^4 = 3^8.$$

If m is odd, then $n = 2^m + 1$ is divisible by 3. One way to see this is to notice that the successive odd numbers differ by 2. Suppose m is odd, and we already know that $2^m + 1$ is divisible by 3. The next number of this type is

$$\begin{aligned} 2^{m+2} + 1 &= (2^m \cdot 2^2) + 1 \\ &= (4 \cdot 2^m) + 1 \\ &= ((3 + 1) \cdot 2^m) + 1 \\ &= (3 \cdot 2^m) + (1 \cdot 2^m) + 1 \\ &= (3 \cdot 2^m) + (2^m + 1). \end{aligned}$$

Since both numbers in this sum are divisible by 3, then so is the sum. Therefore, if $2^m + 1$ is divisible by 3, then so is $2^{m+2} + 1$.

In other words, if the statement is true of one odd number m, then it is also true of the next one. But it is true for $m = 1$ ($2^1 + 1 = 2 + 1 = 3$, which is divisible by 3). Therefore it is true for $m = 3$. Therefore it is true for $m = 5$, etc.

It is, then, sufficient to test $n = 2^m + 1$ for primeness only when m is even.

Discussion of Exercises 3 and 4

3. When $k = 1$ and $m = 4$, we have $n = 2^4 + 1 = 17$ and

$$3^{\frac{17-1}{2}} + 1 = 3^8 = 6562.$$

Since 17 divides 6562, we see that 17 must be a prime according to the test, and it is! When $m = 6$, we have $n = 2^6 + 1 = 65$ and

$$3^{\frac{65-1}{2}} + 1 = 3^{32} + 1,$$

which is not divisible by 65. According to the test 65 is not a prime, which also checks. If you divide $3^{32} + 1$ (a sixteen digit number) by 65, the remainder is 62.

4. If $k = 5$ and $m = 2$ or 4, then n is divisible by 3. If $k = 5$ and $m = 3$, then $n = 41$ and

$$3^{\frac{41-1}{2}} + 1 = 3^{20} + 1,$$

which is divisible by 29. If $k = 7$ and $m = 4$, then $n = 113$ and

$$3^{\frac{113-1}{2}} + 1 = 3^{56} + 1,$$

which is a number of 27 digits.

It will probably be too much work for the children to divide this number by 113 directly. A better way is this. First calculate

$$3^5 = 243$$

and divide by 113. The remainder is 17. Therefore we have

$$3^5 = 17 + (113 \cdot \text{something}).$$

If we square 3^5, we obtain

$$3^{10} = 17^2 + (113 \cdot \text{something}),$$

or

$$3^{10} = 63 + (113 \cdot \text{something}),$$

since the remainder on dividing $17^2 (= 289)$ by 113 is 63. Proceeding in the same way we obtain

$$\begin{aligned} 3^{20} &= 63^2 + (113 \cdot \text{something}) \\ &= 14 + (113 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{40} &= 14^2 + (113 \cdot \text{something}) \\ &= 83 + (113 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{50} &= 3^{10} \cdot 3^{40} = (63 + \cdots)(83 + \cdots) \\ &= (63 \cdot 83) + (113 \cdot \text{something}) \\ &= 31 + (113 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{55} &= 3^5 \cdot 3^{50} \\ &= (17 \cdot 31) + (113 \cdot \text{something}) \\ &= 75 + (113 \cdot \text{something} \end{aligned}$$

and

$$3^{56} = 225 + (113 \cdot \text{something}).$$

But 225 + 1 is divisible by 113 so that the test works!

This calculation is based on the equation

$$(a + c)(b + d) = (a \cdot b) + (a \cdot d) + (b \cdot c) + (c \cdot d).$$

If c and d are divisible by 113, then so is the sum of the last three products on the right.

The same method can be applied to other problems, in particular to the preceding exercises. Thus to test $3^{20} + 1$ for divisibility by 41, we may calculate as follows:

$$\begin{aligned} 3^4 &= 81 = 40 + (41 \cdot \text{something}), \\ 3^8 &= 40^2 + (41 \cdot \text{something}), \\ &= 1600 + (41 \cdot \text{something}), \\ &= 1 + (41 \cdot \text{something}), \\ 3^{16} &= 1^2 + (41 \cdot \text{something}), \\ &= 1 + (41 \cdot \text{something}), \end{aligned}$$

and

$$\begin{aligned} 3^{20} = 3^4 \cdot 3^{16} &= (40 \cdot 1) + (41 \cdot \text{something}) \\ &= 40 + (41 \cdot \text{something} \end{aligned}$$

so that $3^{20} + 1$ is a multiple of 41 since 40 + 1 is divisible by 41.

At the rate of a thousand divisions per second, it would take about $3 \cdot 10^{280}$ years to perform 10^{291} divisions.

5. In order to estimate 2^m yourself, you may use the fact that

$$\log_{10} 2 = .30103,$$

approximately, so that with only a small error

$$2^{3217} = 10^{(.30103)\cdot 3217}$$
$$= 10^{96841},$$

which is a number of 969 digits in decimal notation. Similarly the number of digits in $2^m - 1$ is 667 when $m = 2281$, and 664 when $m = 2203$.

You can astound the children by calculating mentally, using .3 as an approximation for the logarithm of 2. You may look at a number like

$$3^{3217} - 1$$

for a few seconds, and say casually, that this number has about 966 digits $((.3)\cdot 3217 = .965.1)$. Similarly you may use the approximations

$$\log_{10} 3 = .47712 = .5, \text{ approximately},$$
$$\log_{10} 5 = .69897 = .7, \text{ approximately},$$
$$\log_{10} 7 = .84510 = .8, \text{ approximately}.$$

Thus 5^{1000} is a number of about 700 digits since $(.7)\ 1000 = 700$.

By using the method in the text, the children might proceed as follows:

$$2^{10} > 10^3,$$

therefore

$$2^{3217} > 2^{3210} = (2^{10})^{321}$$
$$> (10^3)^{321} = 10^{963},$$

so that when $m = 3217$, n has at least 964 digits. On the other hand, we have

$$2^{13} < 10^4,$$

therefore

$$2^{3217} = 2^{3211}.2^6$$

$$= (2^{13})^{247}.2^6$$

$$< (10^4)^{247}.64$$

$$< 64.10^{988},$$

so that n has at most 990 digits.